rection term. Using this new formulation eliminates the need to neglect modal damping terms, and also facilitates an understanding of the correction made to the mode displacement calculation when using a mode acceleration approach. A simple example has illustrated these results.

Acknowledgments

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Design of a Modalized Observer with Eigenvalue Sensitivity Reduction

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Introduction

THE modalized observer, which was first proposed by Andry et al.,[1] is based upon the well-known fact that the separation theorem does not extend to the eigenvectors related to the plant and observer. This nonseparation may be taken into account when using eigenstructure assignment to design the observer gain matrix. In particular, the observer eigenvectors can be chosen to minimize the impact of a known mismatch of initial conditions between the observer and plant. This is quite useful in flight control problems where a gust disturbance from straight and level equilibrium flight causes nonzero initial conditions in sideslip angle and angle of attack.

In Ref. 1, a modalized observer was designed for the lateral dynamics of the L-1011 aircraft linearized at some trim condition. A choice of observer eigenvectors was proposed that results in the attenuation of the state estimation error, which is induced by an initial condition mismatch in sideslip angle. However, in this Note we show that the observer in Ref. 1 exhibits a large sensitivity of the observer eigenvalues to parameter variation and/or uncertainty in the aircraft stability and control derivatives. We propose a new method for designing modalized observers that achieve both small eigenvalue sensitivity and attenuation of the estimation error caused by an initial condition mismatch. This new design approach for modalized observers is based on minimizing a cost function, which depends on the norm of the observer modal matrix, the condition number of this modal matrix, and directional information about the initial condition mismatch. We compare this new modalized observer design method with the approach described in Ref. 1 to show the advantages of the new method.

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Design Methodology

Consider the linear time invariant plant described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where x is $(n \times 1)$ y is $(r \times 1)$, and u is $(m \times 1)$. Assume that a full-state feedback matrix K has already been obtained by any one of a number of methods. We complete the design by choosing a full-order observer of the form

$$\dot{z}(t) = (A - LC)z(t) + Ly(t) + Bu(t) \tag{3}$$

where L is the observer gain matrix.

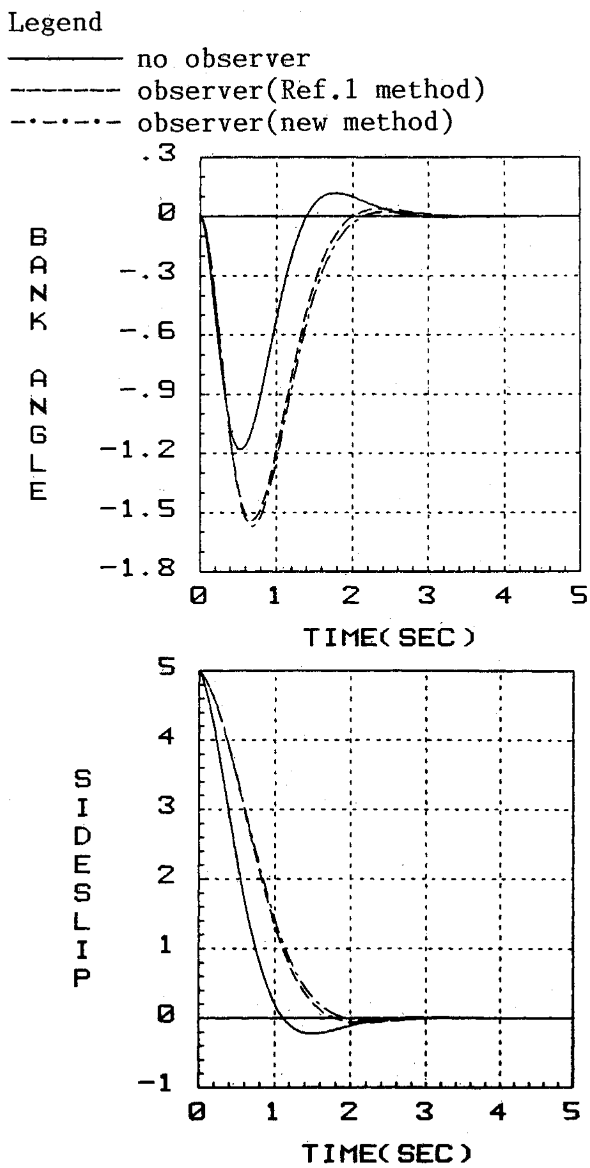

Fig. 1 Aircraft states for three control laws.

Table 1 Observer gains and eigenvalues

	Tuncated observer Gain matrix			Observer eigenvalues using truncated gains	Observer eigenvalues using full machine precision gains
Ref. 1	3.00	−1.00	0.00	$\lambda_1 = -3.000$	$\lambda_1 = -3.000$
Design	25.0	11.0	12.4	$\lambda_2 = -3.707$	$\lambda_2 = -3.500$
Method	−10.5	−3.88	−4.11	$\lambda_{3,4} = -4.131 \pm j\,0.3633$	$\lambda_3 = -4.000$
	−15.0	−5.79	−8.57		$\lambda_4 = -4.500$
New	3.32	0.976	−0.0861	$\lambda_1 = -3.000$	$\lambda_1 = -3.000$
Design	0.118	4.66	−1.11	$\lambda_2 = -3.501$	$\lambda_2 = -3.500$
Method	1.53	−1.18	1.94	$\lambda_3 = -3.999$	$\lambda_3 = -4.000$
	0.352	−1.69	0.0556	$\lambda_4 = -4.500$	$\lambda_4 = -4.500$

The error between the actual and estimated state vector was shown by Kautsky et al.,[2] to be given by

$$e(t) = H\Lambda(t)H^{-1}e(o) \tag{4}$$

where H is the modal matrix of $(A - LC)$, and $\Lambda(t) = \text{diag}(e^{\lambda_1 t},\dots, e^{\lambda_n t})$ with λ_i an eigenvalue of $(A - LC)$. This is easily seen because the error dynamics are described by

$$\dot{e}(t) = (A - LC)e(t)$$

with solution given by

$$e(t) = \exp[(A - LC)t]e(o)$$

Then, Eq. (4) follows by using the relationship $\exp[(A - LC)t] = H\Lambda(t)H^{-1}$.

Andry et al.[1] chose to reduce the impact of the initial condition mismatch $e(o)$ by choosing H^{-1} so as to make $H^{-1}e(o)$ "small." Unfortunately, this also causes the condition number of the modal matrix to be large, which yields an observer that is extremely sensitive to small parameter variations. Here we define the condition number of a matrix to be the ratio of the maximum singular value to minimum singular value.

We propose to design the modalized observer in a new way that will yield an observer that is less sensitive to parameter variations. Observe that the error given by Eq. (4) is bounded as shown below:

$$\|e(t)\| = \|H\Lambda(t)H^{-1}e(o)\| \tag{5}$$

$$\leq \|H\|\cdot\|H^{-1}e(o)\|\cdot\|\Lambda(t)\| \tag{6}$$

$$\leq \|H\|\cdot\kappa_2(H)\cdot\|H^{-1}e(o)\|\cdot\|\Lambda(t)\| \tag{7}$$

$$\leq [\kappa_2(H)]^2\cdot\|e(o)\|\cdot\|\Lambda(t)\| \tag{8}$$

where $\|\cdot\|$ is the 2-norm. To obtain Eq. (7) from Eq. (6), we used $\kappa_2(H) \geq 1$, as shown in Ref. 2. Also, Eq. (8) follows from Eq. (7) upon using $\|H\|\cdot\|H^{-1}e(o)\| \leq \|H\|\cdot\|H^{-1}\|\cdot\|e(o)\|$ and $\kappa_2(H) = \|H\|\cdot\|H^{-1}\|$ where $\kappa_2(H)$ is the condition number of the modal matrix.

We conclude that if no directional information is available about the initial error, then the best strategy would be to minimize the condition number of the modal matrix, as suggested in Ref. 2. We remark that, as shown in Ref. 2, the quantity $\kappa_2(H)$ is related to an upper bound on the sensitivities of the eigenvalues. However, for many problems in flight control, Ref. 1 shows how directional information about $e(o)$ might be available. Since $z(t) = x(t)$ for large t, assume that the composite system (plant plus observer) is in "equilibrium," i.e., $e(t) = x(t) = z(t) = 0$. Perturb the system (not the observer) an amount x_0. Assuming $t = 0$ at the time of perturbation, we have[1]

$$x(o) = x_0 = e(o)$$

Now consider the longitudinal axis or the lateral axis equations for an aircraft. The state vectors may be given as

$$x = \begin{bmatrix} u \\ q \\ \theta \\ \alpha \end{bmatrix} \begin{matrix} \text{perturbed air speed} \\ \text{pitch rate} \\ \text{pitch angle} \\ \text{angle of attack} \end{matrix}$$

$$x = \begin{bmatrix} \phi \\ p \\ r \\ \beta \end{bmatrix} \begin{matrix} \text{bank angle} \\ \text{roll rate} \\ \text{yaw rate} \\ \text{sideslip angle} \end{matrix}$$

From an equilibrium "position" of straight and level flight, all state components are zero. A gust disturbance causes a nonzero initial condition which, in either case, will be of the form[1]

$$x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ c \end{bmatrix} = e_0$$

where c is a constant. With this knowledge, we propose to compute the observer gains by minimizing Eq. (7). This optimization is performed over the eigenvectors, which are constrained to lie in certain subspaces once the eigenvalues are chosen. Thus, with the eigenvalues chosen, minimizing Eq. (7) is equivalent to minimizing F given by

$$F = \|H\|\cdot\kappa_2(H)\cdot\|H^{-1}e(o)\| \tag{9}$$

Upon dividing Eqs. (6–8) by $\|\Lambda(t)\|$ we obtain

$$\|H\|\cdot\|H^{-1}e(o)\| \leq \|H\|\cdot\kappa_2(H)\cdot\|H^{-1}e(o)\| \tag{10a}$$

and

$$\|H\|\cdot\kappa_2(H)\cdot\|H^{-1}e(o)\| \leq [\kappa_2(H)]^2\cdot\|e(o)\| \tag{10b}$$

We observe from Eq. (10a) that minimizing Eq. (9) has the effect of minimizing an upper bound on $\|H\|\cdot\|H^{-1}e(o)\|$.

Also, we observe from Eq. (10b) that minimizing Eq. (9) has the effect of minimizing a lower bound on $\kappa_2(H)$. The condition number of the modal matrix $\kappa_2(H)$ is related to an upper bound on the eigenvalue sensitivities, while $\|H\| \cdot \|H^{-1}e(o)\|$ is related to the tightest bound available for the norm of the estimation error as shown by Eq. (6). Hence, this new design approach should yield an observer that exhibits both small eigenvalue sensitivity and good attenuation of the error induced by an initial condition mismatch of known direction. Finally, we remark that minimizing $\kappa_2(H)$ will not, in general, yield good error attenuation. Conversely, minimizing $\|H\| \cdot \|H^{-1}e(o)\|$ will not, in general, yield small eigenvalue sensitivity.

Example

We consider the linearized lateral dynamics of the L-1011 aircraft as described in Ref. 1. The state variables are bank angle, roll rate, yaw rate, and sideslip angle. The inputs are rudder deflection and aileron deflection.

The control law $u = -Kx$ is given by

$$K = \begin{bmatrix} 0 & 0 & -0.689 & 4.56 \\ -13.1 & -3.13 & 0 & 0 \end{bmatrix}$$

In Ref. 1, the authors assumed that only bank angle and roll rate may be measured. Here we assume that yaw rate is also available for measurement. The observer eigenvalues are chosen, as in Ref. 1, to be $\lambda_1 = -3$, $\lambda_2 = -3.5$, $\lambda_3 = -4$, and $\lambda_4 = -4.5$.

To reflect a gust-induced initial sideslip, we chose the initial state vector to be $x(o) = [0, 0, 0, 5]^T$. However, the observer state is initialized to be the zero vector. Thus, an initial condition mismatch exists in sideslip. The observer gain matrix obtained from using the approach of Ref. 1 is truncated to three significant digits and shown in Table 1 along with the eigenvalues. Observe the extreme sensitivity in the eigenvalues that occurs under the extremely small perturbation caused by truncating the observer gains. This implies that the eigenvalues of $(A - LC)$ will also be sensitive to variations in the elements of the matrix A. We compute a new observer gain matrix by minimizing F in Eq. (9). First, the eigenvalues are chosen that determine the subspaces in which each of the eigenvectors must lie. These subspaces are parameterized by a set of 12 scalars that are initialized at their values from the design based upon the approach of Ref. 1. Then the IMSL, Inc., subroutine ZXMIN[3] is used to perform a quasiNewton search with numerically evaluated derivatives. The optimal observer gain matrix is truncated to three significant digits and shown in Table 1 along with the eigenvalues. Observe the insensitivity of the eigenvalues as compared to the previous design.

The time responses of the actual aircraft states are shown in Fig. 1 for three different control laws. These controllers include 1) feedback of the actual states, 2) feedback of estimated states using the observer based upon the approach of Ref. 1, and 3) feedback of estimated states using the new modalized observer. Observe that, for this example, both observers achieve approximately identical estimation error properties. However, the new observer exhibits significantly smaller eigenvalue sensitivity than the previous observer design.

Conclusion

We have proposed an approach for the design of modalized observers that yields an observer with both good error attenuation to an initial condition mismatch of known direction and small eigenvalue sensitivity. This new design method is based on minimizing a cost function that depends on the norm of the observer modal matrix, the condition number of this modal matrix, and directional information about the initial condition mismatch. An example of the lateral dynamics of the L-1011 aircraft is presented, which shows that this new approach yields a better observer design than a previous design method for modalized observers.

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Simple Hybrid Search Technique for Finding Conic Solutions to the Intercept Problem

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Nomenclature

a = semimajor axis
e = eccentricity
f = $1 - [|r_T(\tau + \Delta\tau)|/p](1 - \cos \Delta\nu)$
G = $[|r_T(\tau + \Delta\tau)|\,|r_S(\tau)|/\sqrt{\mu p}] \sin \Delta\nu$
g = gravitational acceleration at the Earth's surface
I_{max} = p-iteration control parameter (limit)
i = inclination
J_{max} = $\Delta\tau$ prediction control parameter (limit)
k = $|r_T(\tau + \Delta\tau)|\,|r_S(\tau)|(1 - \cos \Delta\nu)$
l = $|r_T(\tau + \Delta\tau)| + |r_S(\tau)|$
M = mean anomaly
m = $|r_T(\tau + \Delta\tau)|\,|r_S(\tau)|(1 + \cos \Delta\nu)$
n_X = nonzero, positive integer control parameter for X_{nom}
p = $a(1 - e^2)$
$r_S(\tau)$ = inertial source position vector at time τ
$r_T(\tau)$ = inertial target position vector at time τ
v = $|\dot{r}_S(\tau_L) + \Delta V|$
X_{nom} = $\eta(1/n[\min\{X\}_1 + \max\{X\}_1])$, where $X \epsilon \{\Delta\tau, |\Delta V|\}$
$\{Y\}_N$ = Nth level search set for Y, where $Y \epsilon \{\tau_L, \Delta\tau, |\Delta V|\}$
γ = local horizontal flight-path angle
ΔE = change in eccentric anomaly equivalent to $\Delta\nu$
ΔF = $-i\Delta E (i = \sqrt{-1})$
ΔV = initial velocity impulse applied at source at τ_L
$\Delta\nu$ = $\cos^{-1}\{[r_T(\tau_L + \Delta\tau) \cdot r_S(\tau_L)]/[|r_T(\tau_L + \Delta\tau)|\,|r_S(\tau_L)|]\}$
$\Delta\tau$ = time of flight from launch to intercept
ϵ_X = $\{X\}_1$ membership tolerance
$\eta(x)$ = $x \epsilon \{X\}_1$ if $\{X\}_L = [X]_1$; otherwise $x_j \notin \{X\}_1$, where x_j is the nearest neighbor of x
μ = gravitation constant
ν = true anomaly
Σ_N = Nth level solution search space $= \{\tau_L\}_N \times \{\Delta\tau\}_1 \times |\Delta V|\}_1$
τ_L = time of launch
Ω = right ascension of the ascending node
ω = argument of perigee

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